Short Papers

Effective Impedance of a Load Filling a Circumferential Slot in a Coaxial Transmission Line

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Abstract —A terminal load uniformly filling a narrow slot placed in the outer conductor of a coaxial transmission line presents a certain effective impedance to TEM waves propagating in the line. The relationship between this value of impedance and the voltage-to-current ratio at this load is established. The two are not necessarily equal. A numerical example is considered.

I. Introduction

Occasionally, the load terminating a sinusoidally driven coaxial transmission line is uniformly distributed around a circumferential electrically narrow band located in the outer conductor of the system. Such an arrangement is shown in Fig. 1. The nominal impedance of the load, assumed here to be known *a priori*, is naturally defined as $Z_A = V_A/I_A$, where V_A, as shown in Fig. 1, is the phasor voltage across the load, while I_A is the phasor current flowing into it. This impedance might represent the loading caused by a radiating slot cut in the outer conductor of a coaxial line in order to form an antenna, as shown in Fig. 2. A boundary value solution of the fields exterior to the line could yield Z_A. Gaps are also sometimes placed in the outer conductor of a coaxial line to form a leaky feeder communication system, as described, for example, by Hill and Wait in [1] and [2].

Elementary transmission theory establishes that, for a lossless transmission line of characteristic impedance Z_0, the impedance observed at a distance L from a terminating load of impedance Z_T is

$$Z(L) = Z_0 \frac{Z_T + jZ_0 \tan(kL)}{Z_0 + jZ_T \tan(kL)} \tag{1}$$

where k is the phase constant of the line. The impedance Z_A of the circumferential load cannot, in general, be substituted for Z_T in the above formula since Z_A is not the impedance which the load presents to transmission line mode (TEM) waves. The field surrounding the circumferential load is a complicated one involving not only the TEM mode but higher order evanescent TM modes as well.

II. Analysis

To determine the relationship between $Z(L)$ and Z_A, we consider a section of transmission line of length L, shown schematically in Fig. 3. The right end of this section is connected to a voltage source and a series load Z_0 matched to the transmission line. The left end of this section is identical to the load terminating the line in Fig. 1. The voltage appearing across Z_A is $V_1 = -I_1 Z_A$. Combining this with the standard equations for a linear passive two-port network, we have

$$-\frac{V_1}{Z_A} = I_1 = Y_{11}V_1 + Y_{12}V_2 \tag{2a}$$

$$I_2 = Y_{21}V_1 + Y_{22}V_2. \tag{2b}$$

Solving these simultaneously, we find

$$\frac{V_2}{I_2} = Z_2 = \frac{1}{Y_2} \tag{3}$$

where

$$Y_2 = Y_{22} - \frac{Y_{12}Y_{21}Z_A}{1 + Z_A Y_{11}}. \tag{4}$$

The impedance Z_{fg} (see Fig. 3) seen looking left into the line between terminals f and g is given by $Z_2 - Z_0$. A knowledge of Y_{22}, Y_{11}, and $Y_{12} = Y_{21}$ will yield Z_{fg}.

To determine these coefficients, we consider the infinite coaxial transmission line shown in Fig. 4. The generator should be regarded as distributed throughout a gap in the outer conductor described by $r = b$, $|x| \leq d$. The electric field in this gap is assumed uniform and is given by $E_x = -V_0/2d$. Recall that d is the width of the gap in Fig. 1.

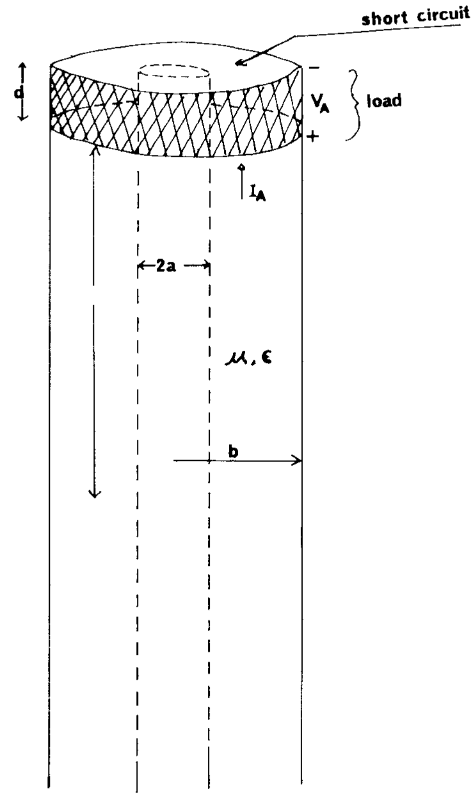

Fig. 1. Configuration of the problem.

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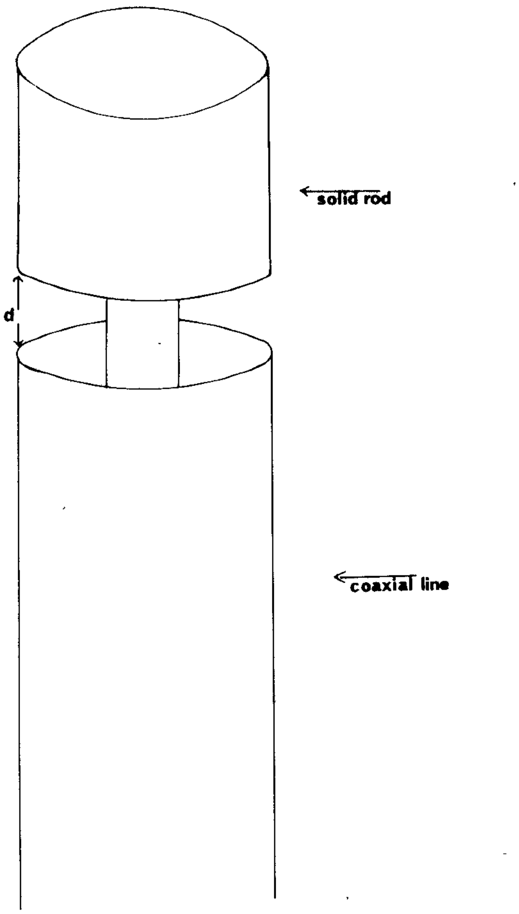

Fig. 2. An arrangement creating a circumferential load.

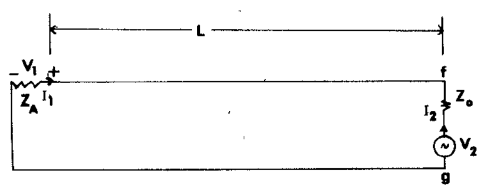

Fig. 3. A scheme for analyzing the problem.

A similar configuration has been considered by Schelkunoff [3], who places the generator in a gap in the inner conductor, $r = a$. We may modify his solution to solve the present problem. The result, with some changes from Schelkunoff's notation, is that the phasor current flowing on the inner surface of the conductor, $r = b$, is

$$I_x(x) = \frac{jkbV_0}{\sqrt{\frac{\mu}{\epsilon}}}\int_{-\infty}^{+\infty}\frac{\sin(\zeta d)}{\zeta d}\frac{1}{(k^2-\zeta^2)^{1/2}}\frac{n_0(\gamma)}{d_0(\gamma)}e^{-j\zeta x}\,d\zeta \quad (5)$$

where

$$n_0(\gamma) = J_1(\gamma b)N_0(\gamma a) - J_0(\gamma a)N_1(\gamma b) \quad (6a)$$

$$d_0(\gamma) = J_0(\gamma b)N_0(\gamma a) - J_0(\gamma a)N_0(\gamma b) \quad (6b)$$

$k = 2\pi/\lambda = \omega\sqrt{\mu\epsilon}$ is the phase constant for the transmission line, $\gamma = (k^2 - \zeta^2)^{1/2}$, and J_n and N_n are Bessel and Neumann functions of order n.

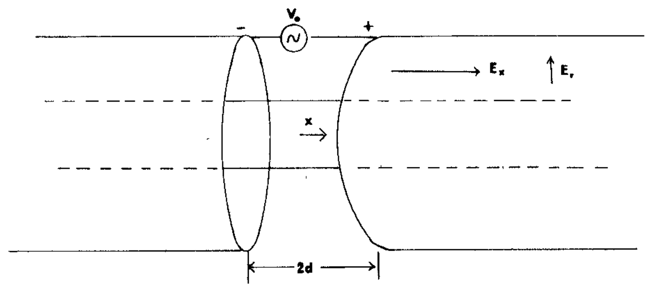

Fig. 4. Infinite coaxial line driven in the outer conductor.

The voltage between the conductors, defined as an integral of the radial electric field:

$$V(x) = \int_b^a E_r(r,x)\,dr$$

is found to be

$$V(x) = \frac{V_0}{2}\frac{\sin(kd)}{kd}e^{-jkx}, \qquad x \geqslant d. \quad (7)$$

Since we assume $d/\lambda \lll 1$, or $kd \ll 1$ (the gap is electrically very narrow), we see from (7) that a voltage wave

$$V(x) = \frac{V_0}{2}e^{-jkx}, \qquad x \geqslant d \quad (8)$$

is radiated outward from the source.

To evaluate $I_x(x)$ in (5) we employ residue calculus. Using small argument expansions for the Bessel functions in (6), we find that the integrand has a simple pole for $\zeta = \pm k$. The residue at k makes a contribution to the current wave moving in the positive x direction of

$$I_{kx}(x) = \frac{V_0}{2}\frac{e^{-jkx}}{Z_0} \qquad \text{for } x \geqslant d. \quad (9)$$

Here $Z_0 = (1/2\pi)\sqrt{\mu/\epsilon}\,\ln(b/a)$ is the characteristic impedance of the transmission line. Again, we have used $kd \ll 1$. The residue at $-k$ leads to a similar expression valid for $x \leqslant -d$.

The remaining portion of the current is associated with all the poles of n_0/d_0 in (5) except those at $\zeta = \pm k$. If the outer radius b of the coaxial line satisfies $kb < 2.40$, then all these poles lie on the imaginary axis in the complex ζ plane. The contribution to the current from the residue at each pole is an attenuated wave. Considering only $x \geqslant d$, we are concerned only with those residues at poles on the negative imaginary axis. Calling these poles ζ_n ($n = 1, 2, \cdots, \infty$), the total contribution to the current is

$$I_{ex}(x) = \sum_{n=1}^{\infty}\frac{2\pi kbV_0}{\sqrt{\frac{\mu}{\epsilon}}}\frac{\sin(d\zeta_n)}{d\zeta_n^2}\frac{n_0(\gamma_n)}{D_0(\gamma_n)}e^{-j\zeta_n x}, \qquad x \geqslant d \quad (10)$$

where

$$\zeta_n = -j\sqrt{\gamma_n^2 - k^2}. \quad (11)$$

The quantity γ_n is a solution of

$$J_0(\gamma_n b)N_0(\gamma_n a) - J_0(\gamma_n a)N_0(\gamma_n b) = 0 \quad (12)$$

and

$$D_0(\gamma_n) = b[J_1(\gamma_n b)N_0(\gamma_n a) - J_0(\gamma_n a)N_1(\gamma_n b)] + a[J_0(\gamma_n b)N_1(\gamma_n a) - J_1(\gamma_n a)N_0(\gamma_n b)]. \quad (13)$$

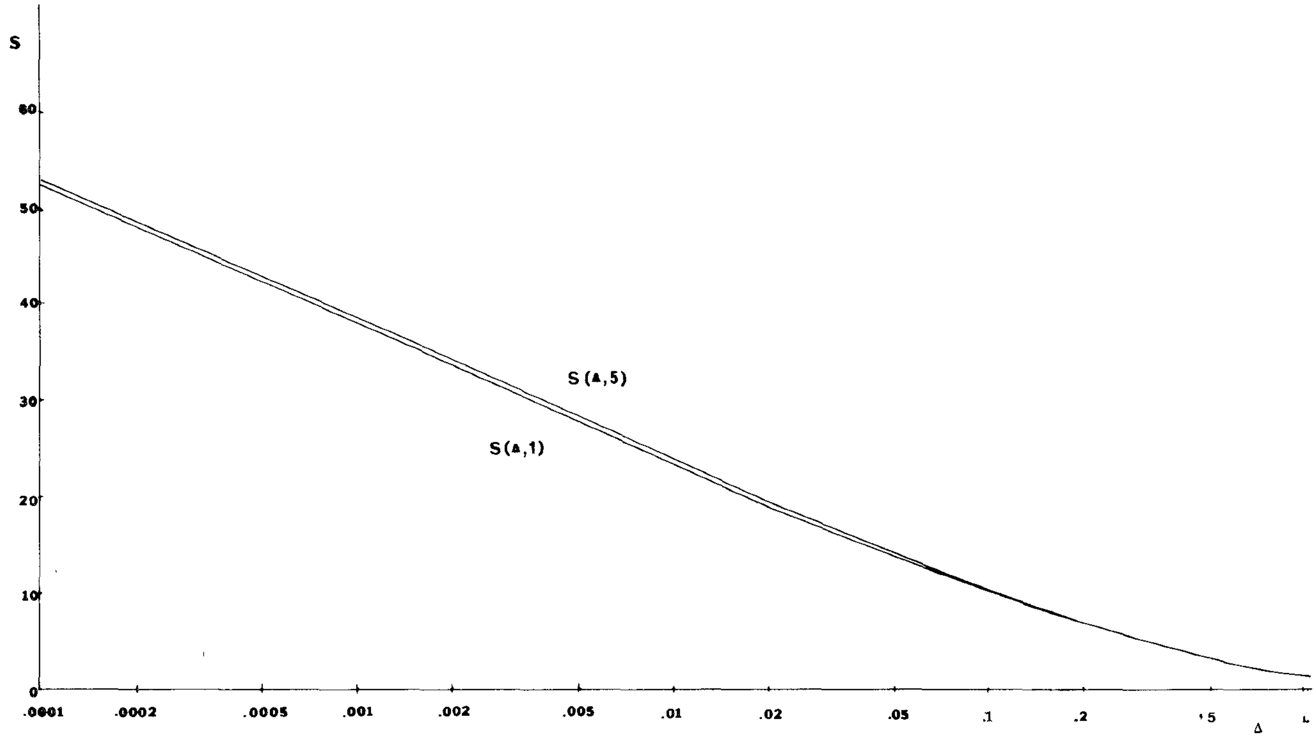

Fig 5. Graph of the function $S(\Delta, c)$ versus Δ for $c = 5$ and $c = 1$

The roots of (12) have been tabulated [4]. A very good approximation, when $1 < b/a \leqslant 5$, is that

$$\gamma_n \doteq \frac{n\pi}{b-a}, \qquad n = 1, 2, \cdots \tag{14}$$

so that

$$\zeta_n \doteq -j\sqrt{\frac{n^2\pi^2}{(b-a)^2} - k^2}. \tag{15}$$

Typically, $kb \ll 1$. Thus, (15) simplifies to

$$\zeta_n = -j\frac{n\pi}{b-a} \tag{16}$$

which, when substituted in (10), reveals a series of highly attenuated waves varying as $e^{-n\pi x/(b-a)}$. The total current on the inside of the outer conductor of the coaxial line is

$$I_x(x) = I_{kx}(x) + I_{ex}(x), \qquad x \geqslant d \tag{17}$$

where I_{kx} and I_{ex} are defined in (9) and (10).

Returning to the problem of finding Y_{11}, Y_{22}, and Y_{12} of (4), we notice, referring to Fig. 3 and equation (2), that $Y_{22} = (I_2/V_2)|_{V_1=0}$. From elementary transmission line theory, we find

$$Y_{22} = \frac{1}{Z_0}\frac{1}{1 + j\tan(kL)} \tag{18}$$

provided $kd \ll 1$ and $L \gg d$.

Let the load Z_A in Fig. 3 (and Fig. 1) be replaced by a voltage source V_1. We now have $Y_{21} = (I_2/V_1)|_{V_2=0}$. Our discussion of the infinite line in Fig. 4 showed that the portion of the current $I_{ex}(x)$ in (17) is highly attenuated. Thus, the load Z_0 in Fig. 3 would, provided L exceeds roughly $2(b-a)$, be exposed only to the transmission line mode of propagation.

With the load Z_A in Fig. 3 and 1 replaced by generator V_1, an application of the method of images yields a configuration electrically equivalent to Fig. 4. Notice that $V_0 = 2V_1$. Using this in (8) and the definition of Y_{21}, we obtain

$$Y_{21} = \frac{-e^{-jkL}}{Z_0} = Y_{12}. \tag{19}$$

To obtain $Y_{11} = (I_1/V_1)|_{V_2=0}$ we again employ the method of images and the configuration of Fig. 4. We have

$$Y_{11} = \frac{I_x(d)}{\frac{1}{2}V_0} \tag{20}$$

where I_x is found from (17), (10), and (9). Thus,

$$Y_{11} = \frac{1}{Z_0} + jB_g \tag{21}$$

where

$$B_g = \sum_{n=1}^{\infty}\sqrt{\frac{\epsilon}{\mu}}\,4kb\frac{(b-a)^2}{\pi d}\frac{\sinh\left(\frac{n\pi d}{b-a}\right)}{n^2}\frac{n_0(\gamma_n)}{D_0(\gamma_n)}e^{-\frac{n\pi d}{b-a}}. \tag{22}$$

We have used (16) and taken $kd \ll 1$.

The substitution of (21), (19), and (18) into (4) yields

$$Z_2 = Z_0\left[\frac{\dfrac{Z_A}{1+jZ_AB_g} + jZ_0\tan(kL)}{Z_0 + j\dfrac{Z_A}{1+jZ_AB_g}\tan(kL)}\right] + Z_0. \tag{23}$$

This is the impedance seen by the generator V_2 in Fig. 3. Subtracting Z_0 from Z_2, we obtain Z_{fg}, the impedance seen between terminals f and g when looking to the left. A comparison of Z_{fg} with (1) shows that the load terminating the left end of the transmission line in Fig. 3 is not simply Z_A but $Z_A/(1 + jB_gZ_A)$, which is the impedance of Z_A in parallel with the admittance jB_g defined in (22). This admittance is associated with the evanescent modes surrounding the gap of width d.

For numerical calculation of B_g, we rewrite (22) as

$$B_g = \frac{2}{\pi} \frac{kb}{\sqrt{\frac{\mu}{\epsilon}}} S(\Delta, c) \tag{24}$$

where

$$\Delta = d/(b-a), \qquad c = b/a \tag{25a}$$

and

$$S(\Delta,c) = \sum_{n=1}^{\infty} \frac{(1-e^{-2n\pi\Delta})(c-1)[J_1(\gamma_n b)N_0(\gamma_n a) - J_0(\gamma_n a)N_1(\gamma_n b)]}{n^2\Delta\left(c[J_1(\gamma_n b)N_0(\gamma_n a) - J_0(\gamma_n a)N_1(\gamma_n b)] - [J_1(\gamma_n a)N_0(\gamma_n b) - J_0(\gamma_n b)N_1(\gamma_n a)]\right)}. \tag{25b}$$

The approximate values (16) and (14) are used.

Fig. 5 shows the results of a digital computer evaluation of $S(\Delta,c)$ for $c=5$. If $c=b/a \to 1$ (with Δ kept finite), then $S(\Delta,c)$ tends to

$$S(\Delta,1) = \sum_{n=1}^{\infty} \frac{1-e^{-2n\pi\Delta}}{n^2\Delta} = \frac{1}{\Delta}\left[\frac{\pi^2}{6} - \sum_{n=1}^{\infty} \frac{e^{-2n\pi\Delta}}{n^2}\right]. \tag{26}$$

An approximate evaluation of this series can be made, for various Δ, on a programmable pocket calculator. $S(\Delta,1)$ has been plotted on Fig. 5. Notice how close in value are the curves $S(\Delta,1)$ and $S(\Delta,5)$. Other curves for $S(\Delta,c)$, where $1<c<5$, lie in the narrow band between the two curves shown in Fig. 5. Because of the insensitivity of $S(\Delta,c)$ to variations in b/a, it is a reasonable approximation to use $S(\Delta,1)$ for $S(\Delta,c)$ whenever $1<b/a\leqslant 5$.

For $\Delta \leqslant 0.01$, there is a useful asymptotic expression for (26):

$$S(\Delta,1) \sim 2\pi \ln\left(\frac{e}{2\pi\Delta}\right). \tag{27}$$

The error obtained in using (27) to approximate (25b) is about 1.7 percent when $\Delta = 0.01$, $b/a=5$, and shrinks with shrinking Δ or b/a.

III. Numerical Example

A lossless line has $b=1$ cm and $b/a=3$. For the material between the conductors, $\mu=\mu_0$ and $\epsilon = 2.24\,\epsilon_0$. The characteristic impedance is $Z_0 = 44.0\ \Omega$. Let a load of $Z_A = 44\ \Omega$ be placed in a gap of width d described by $\Delta = d/(b-a) = 0.1$. A computer evaluation of (25b) gives $S(0.1,3) = 10.29$ (note that (26) would yield 9.20). If $f = 600$ MHz, we have, from (24), $B_g = 4.89 \times 10^{-3}$ ℧. This capacitive susceptance in parallel with Z_A yields an effective load impedance of $42.05 - j\,9.05\ \Omega$, which is moderately different from Z_A.

In this example, the width of the gap is 0.067 cm, which is 2/1000 of the wavelength λ in the line. One should not use the procedure described here unless the gap is very narrow compared to λ, as the theory fails to account for any distribution of the load along the line but assumes it to be concentrated at one location.

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Depth of Penetration of Fields from Rectangular Apertures into Lossy Media

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Abstract —**A widely used device for biomedical applications of microwave energy is the dielectric-loaded waveguide operating in the TE_{10} mode. We have calculated the $(1/e)$ energy penetration depth from such antennas, modeled as rectangular apertures radiating into a lossy medium with dielectric properties resembling those of tissue. The results are presented in nondimensional form from which the characteristics of practical antennas can be estimated. Depending on the dielectric properties of the medium and the size of the aperture, the effective penetration depth can be limited by either the aperture size or the plane-wave penetration depth; practical antennas fall between these two extremes. Experimental results confirm the calculations.**

I. Introduction

Several medical applications of microwave energy have been developed that heat tissue (diathermy and hyperthermia) or measure tissue temperature from the microwave energy that is passively emitted from the body (radiometry). The simplest antenna for such purposes (and one that is widely used) is a rectangular waveguide placed against the surface of the body [1]–[3].

An important consideration is the effective depth of heating or sensing. The field patterns in the tissue beneath an aperture can be calculated using well-established theory [4], [5]. However, these calculations are complex, and in discussing such applications investigators frequently cite the plane-wave penetration depth in the tissues. In contrast, the heating or sensing occurs primarily in the near field of the antenna with a sensitivity function that depends strongly on both the antenna geometry and the material properties of the medium. It is apparent that the effective depth of heating or sensing can be far less than the energy penetration depth of plane waves. We report the penetration depth of energy from rectangular apertures in nondimensionalized form that can be easily applied to a variety of situations. This is a generalization of results previously reported by Turner [6].

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